Lecture # 25

Fourier series

Periodic functions of period 2π

The basis of a Fourier series is to represent a periodic function by a trigonometrical series of the form.

$f(x) = A_0+c_1 \sin (x+\alpha_1) + c_2 \sin (2x+\alpha_2) +c_3 \sin (3x+\alpha_3)+\ldots+c_n \sin (nx+\alpha_n)+ \ldots$

where A_0 is a constant term

$c_1, c_2, c_3 \ldots . c_n$ denote the amplitudes of the compound sine terms $\alpha_1, \alpha_2, \alpha_3 \ldots ..$ are constant auxiliary angles.

Each sine term, $c_n \sin (nx + \alpha_n)$ can be expanded thus:

$c_n \sin (nx + \alpha_n) = c_n\{\sin nx \cos\alpha_n+\cos n x \sin\alpha_n\}= (c_n\sin\alpha_n) \cos nx + (c_n\cos\alpha_n) \sin nx$

$= a_n\cos nx + b_n\sin nx$

the whole series becomes.

$$\mathbf{f(x) = A_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}}$$

As we know that $A_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$;

which can be written as in the expanded from

$A_0+(a_1\cos x+b_1\sin x)+(a_2\cos 2x+b_2\sin 2x)+ .... + (a_n\cos nx+b_n\sin nx)+..........$

$A_0+a_1\cos x+a_2\cos 2x+.+a_n\cos nx+.. .. +b_1\sin x+b_2\sin 2x+... b_n\sin nx+.........$

$f(x) = A_0+a_1\cos x+a_2\cos 2x+.+a_n\cos nx+.. ....+b_1\sin x+b_2\sin 2x+... b_n\sin nx+...$

Fourier coefficients

We have defined Fourier series in the form

$$f(x) = A_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}; \quad n \text{ a positive integer}$$

(a) To find A_0, we integrate $f(x)$ with respect to x from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} A_0\, dx + \sum_{n=1}^{\infty} \left\{ \int_{-\pi}^{\pi} a_n \cos nx\, dx + \int_{-\pi}^{\pi} b_n \sin nx\, dx \right\} = [A_0 x]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} \{ 0 + 0 \} = 2A_0\pi$$

$$2A_0 \pi = \int_{-\pi}^{\pi} f(x)\, dx$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = 1/2\, a_0 ; \quad \text{Where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

(b) To find a_n we multiply $f(x)$ by cos mx and integrate from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\cos mx\, dx = \int_{-\pi}^{\pi} A_0 \cos mx\, dx + \sum_{n=1}^{\infty} \left\{ \int_{-\pi}^{\pi} a_n \cos nx \cos mx\, dx + \int_{-\pi}^{\pi} b_n \sin nx \cos mx\, dx \right\}$$

$$\int_{-\pi}^{\pi} f(x)\cos mx\, dx = A_0 \{0\} +$$

$\sum$

$n=1$

$\{a_n (0) + b_n (0)\} = 0$ for $n \neq m$

$= 0 + a_n\pi + 0 = a_n\pi$ for $n = m$

π

π $\therefore$ $\mathbf{a_n = \dfrac{1}{}}$

$\int$ **f(x) cos nx dx**

$-\pi$

(c) To find b_n we multiply f(x) by sinmx and integrate from $-\pi$ to π.

$\pi\pi\pi$ π

$\int f(x)\sin mx\, dx = \int A_0 \sin mx\, dx + \sum^{\infty} \left\{ \int a_n \cos nx \sin mx\, dx + \int b_n \sin nx \sin mx dx \right\}$

$-\pi$ $-\pi$

π

$n=1$ $-\pi$

∞

$-\pi$

$\int f(x)\sin mx\, dx = A_0 \{0\} + \sum \{a_n (0) + b_n(0)\} = 0$ for $n \neq m$

$-\pi$ $n=1$

$= 0 + 0 + b_n\pi = b_n\pi$ for $n = m$

π

π $\therefore$ $\mathbf{b_n = \dfrac{1}{}}$

$\int$ **f(x) sin nx dx**

$-\pi$

Result For Fourier Series

$\frac{1}{2}$ **f(x)= a_0+$\sum_{n=1}^{\infty}$**

π

{$a_n$cosnx+$b_n$sinnx};

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}$

π

(b) $a_n = \frac{1}{\pi}\int_{-\pi}$

(c) $b_n = \frac{1}{}\int$

f(x) dx = 2×mean value of f(x) over a period

f(x) cos nx dx=2×mean value of f(x) cosnx over a period.

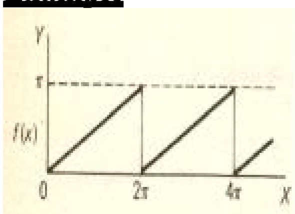
π

f(x) sin nx dx = 2 × mean value of f(x) sin nx over a period.

π $_{-\pi}$

In each case, n = 1, 2, 3, ……

Example



Determine the Fourier series to represent the periodic function shown.

It is more convenient here to take the limits as 0 to 2π.

The function can be defined as

$\frac{}{2}$ f(x) = $\frac{x}{}$

0 < x < 2π

f(x) = f(x + 2π) period = 2π.

Now to find the coefficients

$\frac{1}{}$ $^{2\pi}\frac{1}{}$ $\int^{2\pi}$ x

$+^{2\pi}$

(a) $a_0 = \frac{}{\pi}\int_0$

$a_0 = \pi$

2π

f(x) dx= $\frac{}{\pi}\int_0$

(2) dx = $\frac{}{4\pi}$ [x^2] = π

π(b) $a_n = \frac{1}{}$

$\int_0$ f(x) cos nx dx

2π

$= \int \frac{1}{\pi} \left(\frac{x}{}\right)$ cos nx dx

0
2π 1

1

x sin nx 2π 1π

1

1
$a_n = \frac{1}{2\pi} \int$

x cos nx dx = 2π [n] − n ∫
sin nx dx = 2π { (0 – 0) – n (0)} 0

$\therefore\ a_n = 0$ 2π

0 0

(a) π $b_n = \frac{1}{}$

$\int_0$ f(x) sin nx dx So we now have

2π
1

2π 2π

x b_n =
1 sin nx dx= 1 { [
x cos nx 1
cos nx dx} 1 = −
[2π − 0] = − −

π 0 2

2π − n n

2πn n

00 **$a_0 = \pi$; a_n**

= 0; b_n
= − 1

n Now the general expression for a Fourier series is

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

Therefore in this case

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \{b_n \sin nx\} = \frac{\pi}{2} + \{-\frac{1}{1}\sin x - \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x - ...\} \quad \text{since } a_n = 0$$

$$f(x) = \frac{\pi}{2} - \{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + . \quad \}$$